Observations on Applying the

Dorfman-Steiner Theorem

480-70

Philippe A. Naert *

August 1970

7

*Assistant Professor of Management, Sloan School of Management, M.I.T.

ABSTRACT

Several difficulties are examined concerning the application of the Dorfman-Steiner theorem. In particular, the following points are discussed: How to obtain the optimal value of a variable if its current value is not optimal; the possibility of obtaining suboptimal solutions when a variable which shows little variation is eliminated from the regression equation; it is argued that a possible method around this is to supplement regression analysis by survey methods; the importance on performing a sensitivity analysis on the results; and finally, some reflections on marginal cost equal to marginal revenue as on optimization rule.

536347

Introduction

Dorfman and Steiner developed an optimization rule for a firm whose objective is to maximize profit and whose decision variables are price, advertising and quality [5]. This rule is generally known as the Dorfman-Steiner theorem. In recent years an increasing number of applications of the Dorfman-Steiner theorem have appeared in the professional Journals, Working Papers, and elsewhere.[1] The general procedure used is to estimate the demand function of the product under consideration through single equation regression analysis. Elasticities and marginal products are obtained from the coefficients of the estimated equation, and are compared with what they should be optimally. With this information the firm can then adjust its decision variables in the direction of optimality.

The theorem as presented by Dorfman and Steiner assumes a monopolistic market for the product. More recently, Lambin has made the theorem more general by extending it to the case of an oligopolistic market [8]. Also, the theorem can easily be extended to additional decision variables such as distribution.

Before outlining the content of this paper, some preliminary comments seem in order. First of all, the applications of the Dorfman-Steiner theorem referred to above all assume that causation is unidirectional, that is, advertising expenditures, price and so

on determine sales and not vice versa. Several authors have argued that in fact a system of simultaneous equations is needed.[2] In our discussion it is implicitly assumed that a single equation representation suffices. It is also assumed that the equations presented are free of specification errors.[3]

Advertising in the current period has an effect in this period but also in future periods. Palda and Lambin have made use of Koyck's distributed lags model to take the lagged effect of advertising into account.[4] Again it is assumed here that this is a correct procedure. We should observe though that Montgomery and Silk demonstrated in a recent paper that in a marketing context lags may not be as simple as in the Koyck model and that different elements in the communication mix (e.g. journal advertising, personal selling, product samples, direct mail) may have different lag structures [14].

With these comments in mind we can proceed to an outline of what is to follow. In the first section, we state the Dorfman-Steiner theorem and list the symbols used. In the second section some examples are given of how the optimum value for the price variable can be determined (assuming that the only variable in the model is price). The possibility of obtaining suboptimal values is discussed in the third section. This is the case for example when one decision variable whose value is not optimal is kept constant and is used to determine

the optimal values of other decision variables. The problem of suboptimization is related to the problem of not being able to assess the effect of a given decision variable on sales when the data regarding that variable show little or no variation. It is demonstrated that supplementing regression analysis with survey research may produce additional useful information. In a fourth section we emphasize the importance of performing and reporting a sensitivity analysis of the results. Management wants to know how much profit will increase when the value of a decision variable is altered. It is less interesting to know how much the current value of a decision variable deviates from its optimal value. In the last section some comments are given on equating marginal cost and marginal revenue as a profit maximization rule.

The Dorfman-Steiner Theorem

Let q = quantity sold per period

p = price per unit

c = cost of producing one unit

s = advertising expenditures per period

x = quality index of the product

$\eta_p = \dfrac{\partial q}{\partial p} \dfrac{p}{q}$ = price elasticity

$\mu = p\dfrac{\partial q}{\partial s}$ = marginal revenue product of advertising

$\eta_x = \dfrac{\partial q/\partial x}{\partial c/\partial x} \dfrac{c}{q}$ = quality elasticity

π = profit (not taking into account fixed costs)

The objective of the firm is to maximize

$$(1) \quad \pi = p \, q(p,s,x) - q(p,s,x) \, c\left[q(p,s,x), \, x\right] - s$$

Putting $\partial \pi / \partial p = \partial \pi / \partial s = \partial \pi / \partial x = o$, the first order condition reduces to[5]

$$(2) \quad -\eta_p = \mu = \eta_x \, \frac{p}{c}$$

We will assume that the Hessian determinant is negative definite (unless otherwise stated) so that (1) corresponds to a maximum.[6]

From now on we will assume that $c\left[q(p,s,x), \, x\right]$ is a constant, which we call MC.

Let w equal the percentage of gross margin, that is

$$(3) \quad w = \frac{p - MC}{p}$$

It is easily demonstrated that at optimality

$$(4) \quad -\eta_p = \mu = \eta_x \, \frac{p}{c} = \frac{1}{w}$$

which is simply a restatement of the familiar marginal cost equals marginal revenue rule.

In what follows we will mainly be concerned with price and advertising. This does not mean that quality (or any other decision variable) is unimportant, but considering two decision variables will suffice for the purpose of this paper.

Finding the optimum value for one variable

Three cases will be examined. First of all, quantity demanded is a linear function of price. Secondly, quantity sold is a quadratic function of price. And thirdly, the logarithm of quantity sold is a linear function of the logarithm of price.

First then, suppose that the true relationship between quantity and price is

$$q = a - bp + u$$

where u is a disturbance term, and that the estimated relationship is

$$(5) \quad q = a - bp$$

Suppose that the estimated coefficients are $a = 100$ and $b = 2$ and that they are highly significant. Suppose further that the current price is $30, and that MC = $10 (a constant). The percentage of gross margin is then

$$w = (30 - 10) / 30 = 2/3$$

Let $q_{p=30}$ be the value of q corresponding to a price of $30. From (5) we have

$$q_{p=30} = 100 - 60 = 40$$

Also from (5) we know that

$$\partial q / \partial p = -b = -2$$

So that $\eta_p = -2\frac{30}{40} = -60/40 = -3/2$

From expression (2) the following equality must hold at optimality

(6) $w = \dfrac{1}{-n_p}$ '

With p = 30, w = 2/3, and $1/-n_p$ = 2/3. Therefore, p = \$30 is p*, the optimal price.

Now suppose that the current price is \$25 instead. Percentage of gross margin is then

$$w = (25 - 10)/25 = 3/5$$

and $q_{p=25} = 100 - 50 = 50$

So that $n_p = -2\,\dfrac{25}{50} = -1$

In this case w is not equal to $1/-n_p$ and p = \$25 is not optimal. The optimum price has to satisfy

$$w = \frac{p - MC}{p} = \frac{1}{-n_p}$$

And the optimum price is given by the following expression

(7) $p* = \dfrac{MC}{1 + 1/n_p}$

Now, there is a tendency in practice to substitute some average price elasticity for n_p, that is $n_{\bar{p}} = \dfrac{\partial q}{\partial p}\,\dfrac{\bar{p}}{\bar{q}}$. Suppose that $\bar{p}$ is \$25 and thus $\bar{q}$ = \$50.[7] n_p is then equal to -1 as we found above. Substituting -1 for n_p in (7) we obtain

$$p* = \frac{10}{1 - 1} = \infty$$

This shows in a rather dramatic way that the practice of using an average price elasticity to determine optimal price is to be condemned.[8] The reason is simply that the elasticity is not constant, and that for any value of p different from \$25, η_p will not be equal to -1. Of course, if η_p is a function of p expression (7) is not directly useful. A correct expression for p* is derived below. First note that $\partial q/\partial p$ is constant. If we let q* be the value of q corresponding to p = p*, (6) can be rewritten as

$$(8) \qquad w^* = \frac{1}{-\eta_{p^*}}$$

or

$$\frac{p^* - MC}{p^*} = \frac{1}{\frac{\partial q}{\partial p} \frac{p^*}{q^*}}$$

which reduces to

$$p^* = \frac{q^*}{-\partial q / \partial p} + MC$$

Substituting $q^* = a - bp^*$ and $\partial q/\partial p = -b$, we obtain

$$p^* = \frac{a - bp^*}{b} + MC$$

Therefore

$$p^* = \frac{a}{2b} + \frac{MC}{2}$$

In our case p* is then

$$p^* = \frac{100}{2 \times 2} + \frac{10}{2} = 25 + 5 = 30$$

So, if the current price is \$25, and the estimates of a and b are 100 and 2 respectively, the firm should increase its price by about 20 per cent

if its objective is to maximize profit.[9]

Let us now consider a situation where things get a little more
complicated. Suppose that the true relationship between quantity
demanded and price is as follows

$$q = \alpha - \beta p + \gamma p^2 + u$$

and that the estimated relationship is

$$(9) \quad q = a - bp + cp^2$$

with a = 302, b = 50, c = 1. Also assume that the coefficients are
highly significant. Furthermore MC = \$2, $\bar{p}$ = \$4, and $\bar{q}$ = 118. So
that average price elasticity $\eta_{\bar{p}}$ is

$$\eta_{\bar{p}} = (-b + 2c\bar{p}) \frac{\bar{p}}{\bar{q}} = (-50 + 8) \, 4/118 = -1.424$$

$$w = (4 - 2) / 4 = 1/2$$

Therefore p = \$4 is not optimal. Again, it is incorrect to write

$$p^* = \frac{MC}{1 + 1/\eta_{\bar{p}}} = \frac{2}{1 - 0.702} = \$6.74$$

p* can be obtained by using expression (8). Where the demand function
is given by (9), expression (8) reduces to

$$\frac{p^* - MC}{p^*} = - \frac{a - bp^* + cp^{*2}}{p^* (-b + 2cp^*)}$$

After multiplying through and rearranging terms we obtain

$$(10) \quad 3cp^{*2} - 2p^*(b - cMC) + bMC + a = 0$$

or

$$(11) \quad p^* = (b - cMC) / 3c \pm 1 / 3c \left[(b-cMC)^2 - 3c(bMC + a) \right]^{1/2}$$

Here an additional difficulty arises. What value of p* corresponds to a maximum, the smaller root or the larger root? In order to obtain an answer to that question we need to examine the second order conditions. Profit π is

$$\pi = (p - MC)q = ap - bp^2 + cp^3 - aMC - bpMC + cp^2MC$$

The first order condition for maximization is

$$\partial\pi / \partial p = a - 2bp + 3cp^2 - bMC - 2cpMC = 0$$

which is (10).

For a maximum we need $\partial^2\pi/\partial p^2 < 0$, or

$$- 2b + 3cp + 2cMC < 0$$

which reduces to

$$(12) \qquad p < \frac{b - cMC}{3c}$$

Therefore, the smallest root of expression (11) corresponds to a maximum whereas the larger root corresponds to a minimum. The optimal price is then

$$p^* = (50 - 2) / 3 - 1/3 \left[(50-2)^2 - 3(50\times2 - 302)\right]^{1/2}$$
$$= 16 - 1/3 \left[2,304 - 3(402)\right]^{1/2} = \$4.95$$

and the current price should be increased by about 24 percent. With p = \$4.95, demand is

$$q = 302 - 50(4.95) + (4.95)^2 = 79$$

And profit is

$$\pi = (p - MC)q = 2.95 \times 79 = \$233.05$$

Suppose that the current price is $50. Substituting this in our demand equation (9) we would find $p = 302 - 50 (50) + 50^2 = 302$, and corresponding profit would be $(50 - 2) 302 = \$14,496$. This nonsensical result leads to a further observation. $q_t = a - bp_t + cp_t^2$ is a good fit <u>for the range of observations</u>. $q_t = a - bp_t + cp_t^2$ is shown in Figure 1. From that figure it is clear that the range over which the curve represents a good fit lies certainly within the range of prices o to 7, so that the statement that for a price of $50, demand would be 302 and profit $14,496 is meaningless. Note that if a, b, and c were such that q is positive for all values of p, then the maximum possible relevant range would be o to b/2c, that is, the range in which q is a decreasing function of p.

Finally, suppose that log q is a linear function of log p. The estimated relationship is then

$$(13) \quad \log q = a - b \log p$$

In this case η_p is a constant namely -b, and expression (7) can be applied directly to find the optimal price. Suppose $b = 2$, $MC = \$7$, and the current price is $10. With a price of $10, percentage of gross margin is $w = (10 - 7)/10 = 3/10$, so that $w = 3/10 \neq 1/-\eta_p = 1/2$, and the current price is too low. The optimal price is

$$p^* = \frac{7}{7 - 1/2} = \$14$$

The price should therefore be increased by approximately 40 per cent.

Finding the optimum values of two decision variables.

Suppose that demand is a function of two decision variables, namely price and advertising, and that the estimated functional relationship is as follows[10]

$$(14) \qquad q = a - bp + c \log s$$

Price elasticity is

$$\eta_p = \frac{-bp}{q}$$

At optimality $\quad w^* = 1/-\eta_p \qquad$ or

$$\frac{p^* - MC}{p^*} = \frac{q^*}{bp^*}$$

$$p^* = \frac{q^*}{b} + MC = \frac{a - bp^* + c \log s^*}{b} + MC$$

Solving for p^* we obtain

$$(15) \qquad p^* = \frac{a}{2b} + \frac{c}{2b} \log s^* + \frac{MC}{2}$$

Finding p^* is not as trivial as it was in the previous section, because q^* is not only a function of p^* but also of s^*. However, there is also an optimality condition for advertising which will give us a second expression relating optimal advertising and optimal price. At optimality

$$(16) \qquad w^* = 1/q^*$$

From (14) we have

$$\partial q/\partial s = (\partial q/\partial \log s)(\partial \log s/\partial s) = c/s$$

Therefore, at optimality

$$w^* = s^* / p^* c$$

or

$$\frac{p^* - MC}{p^*} = \frac{s^*}{p^* \, c}$$

Solving for s*, we obtain

(17) $\quad s^* = c(p^* - MC)$

(15) and (17) form a system of two equations in two unknowns. From (17) we find

(18) $\quad p^* = s^*/c + MC$

Substituting (18) for p* in (15) we get

$$\frac{s^*}{c} + MC = \frac{a}{2b} + \frac{c}{2b} \log s^* + \frac{MC}{2}$$

Or

$$s^* = \frac{ac}{2b} + \frac{c^2}{2b} \log s^* - \frac{cMC}{2}$$

For any value of s we can write

(19) $\quad S = s - \frac{ac}{2b} + \frac{cMC}{2b} - \frac{c^2}{2b} \log s$

S will then be zero when s = s*. S as a function of s is shown in Figure 2. For s approaching zero, S approaches infinity and also for s approaching infinity, S approaches infinity. So, assuming that a real optimal value for s exist, there will in fact be two values of s making S equal to zero. We have to check the second order conditions in order to find out which value of s corresponds to the maximum. Profit π is

$$\pi = (p - MC) \, q - s$$

At optimality we know from (18) that $p^* = s^*/c + MC$. Using this, q* can be written as

$$q^* = a - b(s^*/c + MC) + c \log s^*$$

So that profit is

$$\pi = \frac{s^*}{c}\left[a - b(s^*/c + MC) + c\log s^*\right] - s^*$$

$$\frac{d\pi}{ds^*} = \frac{a}{c} - \frac{bs^*}{c^2} - \frac{bMC}{c} + \log s^* - \frac{bs^*}{c^2} + \frac{cs^*}{cs^*} - 1 = 0$$

$$\frac{d\pi}{ds^*} = \frac{a}{c} - \frac{2bs^*}{c^2} - \frac{bMC}{c} + \log s^* = 0$$

which is equivalent to $S = 0$.

The second order condition is

$$\frac{d^2\pi}{ds^{*2}} = -\frac{2b}{c^2} + \frac{1}{s^*} < 0$$

Now comparing S and $\frac{d\pi}{ds^*}$ we see that

$$S = -\frac{c^2}{2b}\frac{d\pi}{ds^*}$$

The slope of S at the maximum is

$$\frac{dS}{ds^*} = -\frac{c^2}{2b}\frac{d^2\pi}{ds^{*2}}$$

Since $d^2\pi/ds^{*2}$ must be negative at the maximum, dS/ds^* must be positive. For the smaller value of s for which $S = 0$, say s^{**}, $dS/ds < 0$ and hence $d^2\pi/ds^2 > 0$, so that s^{**} corresponds to a minimum. While the existence of a minimum may seem strange, it is easily explained if we explore what happens to profit if we let s=o. Expression (15) and (18) combined give us the optimum value for p and s. But for any value of s, (15) gives the optimum value of price for that given value of s. Now from (15) we see that for s approaching o, the optimal price approaches $\frac{c}{2b}\log s$, that is minus infinity, a nonsencical result. Similarly from

(14)   we find that for s approaching zero q approaches

$$- b(\frac{c}{2b} \ \log s) \ + \ c \ \log s \ = \ \frac{c}{2} \ \log s$$

so that q also approaches minus infinity, and therefore profit would

become infinitely large. The reason for obtaining this meaningless

result for s = o is simply that the relationship (14) was not estimated

using values of s between zero and infinity, but using values of s

within a range s^- to s^+ say (see Figure 3). While the relation

between sales and advertising may actually follow a logistic pattern,

the logarithmic representation will still be a very good one in the

range of observations s^- to s^+. But then we should not use (14) in

making assessments about the effect of advertising on sales for values

of s far outside the range s^- to s^+, for example, very small or very

large values of s. The shape of profit as a function of s based on

expression (14) is illustrated in Figure 4 (full line). The dotted

line is the profit function when the true relationship follows a

logistic curve.

We can now proceed to determining s*. Take any value of s such

that S is positive and dS/ds is also positive.[11] Call this value s^o.

The value for s at the next iteration, s^1 is found as follows. From

Figure 2 we see that

$$\frac{dS}{ds}\bigg|_{s=s^o} \ = \ \frac{S(s^o)}{s^o - s^1}$$

Or $$s^1 \ = \ s^o - \frac{S(s^o)}{\frac{dS}{ds}\big|_{s=s^o}}$$

And in general[12]

$$(20) \quad s^{i+1} = s^i - \frac{S(s^i)}{\frac{dS}{ds}\Big|_{s=s^i}} \qquad i=o; \ 1, \ 2, \ldots \ldots$$

And we will say that

$$s^{i+1} = s^* \quad \text{if} \quad S(s^i) \leq \varepsilon, \quad \text{or alternatively,}$$

$$s^{i+1} = s^* \quad \text{if} \quad s^{i+1} - s^i \leq \delta$$

where ε and δ are arbitrarily chosen small positive numbers.[13] This procedure is now clarified by an example. Suppose that the coefficients a, b, and c are estimated to be 100, 2, and 10 respectively and that MC = $6. The demand function (14) can now be written as

$$q = 100 - 2p + 10 \log s$$

S was given by (19) and becomes

$$S = s - \frac{100 \times 10}{2 \times 2} + \frac{10 \times 6}{2} - \frac{10^2}{2 \times 2} \log s$$

$$= s - 220 - 25 \log s$$

Suppose we take $s^o = \$400$. $\log s^o = 5.9913$ and $S(s^o) = 400 - 220 - 149.78 = 30.22$

$$\frac{dS}{ds}\Big|_{s=s^o} = 1 - \frac{25}{400} = 0.938$$

Using (20) we can find s^1

$$s^1 = 400 - \frac{30.22}{0.938} = 400 - 32.21 = 367.79$$

$$\log s^1 = 5.9075$$

$$S(s^1) = 367.79 - 220 - 147.69 = 0.10$$

$$\frac{dS}{ds}\Big|_{s=s^1} = 1 - \frac{25}{367.69} = 0.933$$

$$s^2 = 367.79 - \frac{0.10}{0.933} = 367.68$$

$$\log s^2 = 5.9071$$

$$S(s^2) = 367.68 - 220 - 147.68 = o$$

Therefore

$$s^* = \$367.68$$

Using (18) to obtain p* we find

$$p^* = 367.68/10 \div 6 \overset{\sim}{=} \$42.77$$

Maximum profit is then found as follows

$$q^* = 100 - 85.54 \div 10 \times 5.907 = 73.53$$

Profit is $(p^* - MC)q^* - s^*$ or

$$\pi_{max} = (36.77)\, 73.53 - 367.68 = \$2,335.87$$

Recall that the derivation of s* and p* makes use of the values of a, b, and c obtained from a regression analysis. So in order to have confidence in the values of s* or p*, the estimates of the coefficients over their standard errors, i.e. their t-values should be highly significant. Suppose now that there is little variation in one of the variables, say the price variable. The coefficient of the price variable will be insignificant and may even have the wrong sign.[14] And the coefficient of price obtained from the regression analysis is of no use in trying to establish whether the current price is close to the optimal price or not. The price variable is then dropped from the equation. This does not mean that price has no effect on demand, but that its effect is approximately constant, and will show up in the constant term of the equation. The optimum level of advertising expenditures is obtained from (17) where current price or average price is used as a proxy for optimal price. Most authors do this without warning the reader that the

resulting level of advertising expenditures may be suboptimal. Indeed, the practice of taking current price or average price, whichever is used, as a proxy for optimal price, is fine as long as that proxy is approximately equal to the optimal price. If this is not so, solving for s* will still give the optimum level of advertising expenditures for that given price, but from an overall point of view may be suboptimal To illustrate this we consider five values for price: the optimal price, two prices below and two prices above. For each price we compute the optimal level of advertising expenditures and the corresponding profit. We also compute profit for other values of advertising. The results are presented in Table 1 and are illustrated in Figure 5. The analysis contains much detail which will be useful in the section on sensitivity analysis. Table 1 and Figure 5 clearly illustrate that when advertising is optimized for a non optimal price, profit is less then the overall maximum. Overall maximum profit is $2,335.87 and corresponds to a price of $42.77. If the current price is higher, maximum possible profit is lower, e.g. with a price of $50 corresponds a maximum possible profit of $2,237.40. Similarly if price is lower than the optimal price, maximum possible profit is again lower, e.g. with a price of $36 corresponds a maximum possible profit of $2,250.6.

In summary then we can say the following. If there is little or no variation in the price data (or in the data regarding any variable), the effect of price on sales is approximately constant and will be taken up by the constant term in the regression equation, when price as a variable is dropped from the equation. The estimated demand equation is then

$$(21) \qquad q = a' - c \log s$$

where $a' = a - b\bar{p}$, and where $\bar{p}$ is current (or average) price. The

optimal value of advertising is then obtained using $\bar{p}$ as a proxy for

p^*. We demonstrated by example that the further away $\bar{p}$ is from p^*,

the lower the maximum possible profit becomes. So instead of assuming

that $\bar{p}$ is approximately equal to p^* when regression analysis is of

no help in determing the effect of changes in price on sales, we should

take recourse to other methods. One possible method is briefly described below.

Information on the effect of price on sales could be obtained by conducting a

survey. A representative set of consumers who buy product of interest could be

asked how much of the product they would buy for different price levels. Suppose

that our sample consists of j persons N_1, N_2, ...N_j. If price were

equal to p_1, comsumer N_1, states that he would buy q_1^1, N_2 would buy

q_1^2, ..., N_j would buy q_1^j. If the sample contains x per cent of the popula-

tion, q_1, quantity demanded when price equals p_1 would be estimated to be

$$q_1 = \frac{(q_1^1 + q_1^2 + ... + q_1^j)\ 100}{x}$$

Or in general, for a price p_i, quantity demanded would be

$$q_i = \frac{(q_i^1 + q_i^2 + ... + q_i^j)\ 100}{x}$$

A line can then be fitted through the points (p_1, q_1), (p_2, q_2), ...,

(p_i, q_i), ... which is our estimated demand function.[15] Let the

estimated demand curve be

$$(22) \quad q = a'' - bp$$

The question arises what the effect of advertising is in this function. Only prices are varied so that other things remain equal. Thus the effect of advertising is the effect of the current level of advertising expenditures, that is $\bar{s}$. For $p = \bar{p}$ we then have two estimates of the corresponding demand $\bar{q}$. One from (21), namely

$$(23) \quad \bar{q} = a' + c \log \bar{s}$$

and another from (22),

$$(24) \quad \bar{q} = a'' - b\bar{p}$$

The values of $\bar{q}$ from (23) and (24) may be different. We will assume here that the difference is due to the fact that the sample contained only x per cent of the population. Suppose that the difference is Δ so that

$$(25) \quad a' + c \log \bar{s} = a'' - b\bar{p} + \Delta$$

Since a', c, $\bar{s}$, a", b, and $\bar{p}$ are known we can solve for Δ. Comparing (14) and (21) we can write

$$(26) \quad a' = a - b\bar{p}$$

Substituting (26) in (25) we have

$$a - b\bar{p} + c \log \bar{s} = a'' - b\bar{p} + \Delta$$

Solving for a we obtain

$$(27) \quad a = a'' + \Delta - c \log \bar{s}$$

At this point we have estimates of a, b, and c and we can determine p* and S* with the procedure presented earlier in this section.

Sensitivity Analysis

As far as this author knows, none of the studies which make use of the Dorfman-Steiner theorem contains a sensitivity analysis of the results. Yet this is most important from a management point of view. It is one thing to know that current advertising outlays are twenty per cent too low, but it is more interesting to know how the profit picture will change if management decides to spend the optimal amount on advertising. In other words, a sensitivity analysis is called for. For the case presented in the previous section the sensitivity of profit to advertising can be examined from Table 1 and Figure 1. It is seen from these that, for a given price, profit is rather insensitive to changes in the level of advertising expenditures in a broad range about the optimum value. Let us first consider the situation where price is optimal, i. e. $p = \$42.77$. Now suppose that advertising is $265, that is, 28 per cent below the optimal amount. Profit is approximately $2,312 or only 1 per cent below optimal. Similarly, if advertising is $505, or about 30 per cent above the optimum, profit will have dropped by only 1 per cent. Table 2 summarizes similar arguments for the different situations shown in Figure 1 and Table 1 both for one per cent and five per cent below maximum profit.

From Table 2 we observe that large percentage changes in advertising outlays result in only small percentage changes in profit in a broad range of values about the optimum. Profit is most insensitive to

advertising for low prices and becomes slightly more sensitive for larger prices.[16]

So far we analyzed the sensitivity of profit to advertising. We now proceed to the examination of the sensitivity of profit to price. To do this we construct Table 3, which shows optimal advertising outlays corresponding to a given price, and the corresponding profit. The data in Table 3 are then used to construct Figure 6. It is easily seen that profit is much more sensitive to price deviations from the optimal price than it is to deviations from optimal advertising expenditures. For example if price is $36, that is, about 16 per cent below the optimal prices, profit is about 3.7 per cent below optimal. Similarly, if price is $50 or about 17 per cent above the optimal price, profit is 4.2 per cent below optimality. Also the range of values of p for which profit is not very sensitivity is now smaller. For example, if price increased another 20 per cent (from $50 to $60) profit decreases another 21 per cent (from $2,237.40 to $1,776.60). In short then we may say that for the case at hand it is more important for management to know the optimum value of price than it is to know the optimum value of advertising.[17] This reinforces our earlier assertion that if regression analysis does not enable us to measure the effect of a certain variable on sales, other methods should be used to supplement the information obtained by regressing sales and the remaining independent variables.

The reader may well wonder at this point whether the low
sensitivity to advertising is just due to the fact that the example
is a contrived one or whether such low sensitivity could also exist
in a real situation. For that purpose we analyzed the case of a
frequently purchased food product reported by Lambin [10] [11,
Chapter 4]. Lambin obtains the following equation

$$(28) \quad q_t = -32,733 + 12,423 \log I_t + 0.507 q_{t-1} + 1,777 \log s_t$$
$$- 2.2 W_t + 843 \log d_t$$

where

I_t = real private disposable income

W_t = weather: rainfall adjusted for seasonal variations

q_{t-1}= goodwill: lagged dependent variable

s_t = real advertising expenditures per 1,000 potential
consumers

d_t = visit frequency to sales outlets

Note that price does not appear as a variable in equation (28). In
the original equation price was included but was later dropped due to
lack of variation in the price data. The average price of the product
is 6 BF (Belgian francs). Lambin's analysis implicitly assumes that the
price of 6 BF is optimal. Lambin's study of the profitability of
advertising is ceteris paribus, that is all variables except advertising
are fixed.[18] Equation (28) reduces to

$$(29) \quad q_t = k + 1,777 \log s_t$$

Common logarithms were used so that average marginal product of advertising, $\partial q / \partial s$ is

$$\frac{\partial q}{\partial s} = \left(\frac{\partial q}{\partial \log s}\right)\left(\frac{0.4343}{\bar{s}}\right)$$

where $\bar{s}$ is average advertising expenditures. Similarly, average advertising elasticity is

$$\eta_{\bar{s}} = \left(\frac{\partial q}{\partial s}\right)\frac{\bar{s}}{\bar{q}}$$

Lambin does not give $\bar{s}$ and $\bar{q}$ but they can readily be derived from $\partial q/\partial s$ and η_s. $\partial q/\partial s$ is 0.024. From (29) we have $\partial q/\partial \log s = 1,777$, so that

$$\bar{s} = \frac{1,777 \times 0.4343}{0.224} = 3,440 \quad BF$$

η_s is 0.190 and thus

$$\bar{q} = \frac{0.224 \times 3,340}{0.190} = 4,060 \quad BF$$

Now $1,777 \log \bar{s} = 1,777 \log 3,340 = 6,273$

Substituting $\bar{q}$ and $1,777 \log \bar{s}$ in (29) we can find k.

$$k = 4,060 - 6,273 = -2,213$$

A negative value for k may seem strange since it incorporates the contributions to sales of all other variables. We should observe however, that the constant term in (28) is a large negative number, namely $-32,733$.

Percentage of gross margin is also given and is 55 per cent. Optimal advertising expenditures can now be determined as follows. At optimality (16) must hold, that is μ^* must equal $1/w^*$. Since it is implicitly

assumed that the price is optimal, w* is known. Therefore μ^* must equal

$$\mu^* = 1/0.55 = 1.818$$

Or

$$\partial q/\partial s = 1.818 / 6 = 0.303$$

Optimal advertising expenditures are then[19]

$$s^* = \frac{(\partial q/\partial \log s) \ (0.4343)}{\partial q/\partial s}$$

$$= \frac{(1,777) \ (0.4343)}{0.303} = 2,545 \ BF$$

We will now compare profit when advertising outlays are 3,440 BF with profit when advertising outlays are 2,545 BF. First we need the value of MC. We know that $(p - MC) / p = 0.55$, and that $p = 6$ BF. MC is then 2.7 BF. The profit function is

$$(p - MC) \ q - s = 3.3 \ q - s$$

With s = 3,440 BF, we found above that q = 4,060, and the corresponding profit is

$$3.3 \times 4,060 - 3,440 = 9,958 \ BF$$

At optimality, s = 2,545. Since log 2,545 = 3.4057

$$q^* = - 2,213 + 1,777 \times 3.4057 = 3,839$$

Maximum profit is

$$3.3 \times 3,839 - 2,545 = 10,124 \ BF$$

Thus assuming that the firm considers only the short term effects of advertising, we find that whereas actual advertising expenditures are 35 per cent higher than optimal, profit is only 1.5 per cent below its maximum value. So a low sensitivity of profit to advertising is observed.

.

So far we have neglected the long term effects of advertising. It can be shown that the relationship between long term and short term marginal revenue product of advertising is as follows[20]

$$(30) \qquad \mu(LT) \;=\; \frac{\mu(ST)}{1 - \dfrac{\lambda}{1+i}}$$

where $\mu(LT)$, $\mu(ST)$ are long term and short term marginal revenue product of advertising respectively, λ is the retention rate (goodwill), and i represents the time value of money (discount rate).

An estimate of λ is the coefficient of lagged sales, i.e. 0.507.

If the firm takes a long term view, at optimality we must have

$$\mu^*(LT) \;=\; 1/w^* \;=\; 1.818$$

The corresponding value of $\mu(ST)$ is obtained from (30)

$$\mu^*(ST) \;=\; \mu^*(LT)\left(1 - \frac{\lambda}{1+i}\right)$$

With $\lambda = 0.507$ and assuming that $i = 20$ per cent $\mu^*(ST)$ is

$$\mu^*(ST) \;=\; 1.818\left(1 - \frac{0.507}{1+0.20}\right) \;=\; 1.050$$

$\partial q/\partial s$ corresponding to $\mu(ST) = 1.050$

$$\partial q/\partial s \;=\; 1.050/6 \;=\; 0.175$$

And the optimal level of advertising expenditures, s^*_{LT} is

$$s^*_{LT} \;=\; \frac{(1,777)(0.4343)}{0.175} \;=\; 4,410 \;\; BF$$

And $q = -2,213 + 1,777 \log 4,410 = 4,263$

Long term profit π^* (LT) is

$$\pi^*(LT) = q(p - MC) \left[1 + \frac{\lambda}{1+i} + \cdots \cdots \right.$$

$$+ \left. \frac{\lambda^n}{(1+i)^n} + \cdots \cdots \right] - s^*_{LT}$$

$$= \frac{q(p - MC)}{1 - \frac{\lambda}{1+i}} - s^*_{LT}$$

$$= \frac{(3.30)(4,263)}{0.5775} - 4,410 = 19,950 \quad BF$$

Actual advertising expenditures are 3,440 BF, and the corresponding

quantity sold is 4,060, so that long term profit with $s = 3,440$ BF is

$$\pi(LT) = \frac{(3.30)(4,060)}{0.5775} - 3,440 = 19,760 \quad BF$$

Actual advertising expenditures are 22 per cent below the optimal

amount, but long term is only about 1 per cent below optimal.

Therefore in the case presented by Lambin, profit is rather

insensitive to deviations of advertising expenditures from their optimal

value.

Marginal revenue equal to marginal cost as an optimization rule

In this last section we explore whether equating marginal revenue

to marginal cost actually represents maximization of profit. We will

explore this by concentrating on the advertising variable. We will

again consider the short term view first and then the long term view.

From (31) we see that the condition $\mu^* = 1/w^*$ now becomes

(32) $$\mu^* = \frac{1+r}{w^*}$$

Suppose that the opportunity cost of investing in advertising is 20 per cent. In that case $r = 0.20$. Building on the study by Lambin described in the previous section we find

$$\mu^* = 1.2 / 0.55 = 2.1818$$

The corresponding marginal product $\partial q/\partial s$ is

$$\partial q/\partial s = 2.182 / 6 = 0.3638$$

And

$$s^* = (1,777)(0.4343)/(0.3638) = 2,121 \text{ BF}$$

Since actual advertising outlays are 3,440 BF, we observe that if the firm takes a short term view and if the opportunity cost of investing in advertising is 20 per cent, actual advertising expenditures are 62 per cent above their optimum value. Let us now compare the profit figures for $s = 2,121$ BF and $s = 3,440$ BF. In doing so we have to take into account that the difference $3,440 - 2,121 = 1,319$ BF can earn a return of 20 per cent. Profit with $s = 3,440$ BF was found in the previous section to be 9,958 BF. Profit with $s = 2,121$ BF is

$$\pi = 3.3 - 2,2.3 + 1,777 \log 2,121 - 2,121 - (3,440 - 2,121)$$

$$(0.20) = 10,241 \text{ BF}$$

Profit with s optimal is therefore 283 BF or about 3 per cent higher than when $s = 3,440$ BF.

Finally, suppose the firm takes a long term view. $L^*(LT)$
has to be 2.1818. Using (3_0) we find $\mu^*(ST)$.

$$\mu^*(ST) = 2.1818\,(1 - \frac{0.507}{1.20}) = 1.26$$

and

$$\partial q/\partial s = 1.26 \,/\, 6 = 0.21$$

$$s^*_{LT} = (1,777)(0.4343) \,/\, (0.21) = 3,675 \quad BF$$

Therefore if the firm takes a long term view and the opportunity cost
of investing in advertising is 20 per cent, actual advertising expenditures
are only 6.4 per cent below optimum, rather than 22 per cent as was
obtained under the implicit assumption that the opportunity cost is zero.[22]

Summary and Conclusions

In this paper four problems were studied related to applying the
Dorfman-Steiner theorem in deriving optimal values for marketing
decision variables. First of all, we showed how to determine the optimum
value of a particular variable, if that variable is the only active decision
variable. Secondly, we discussed the possibility of suboptimization if
some variable is excluded from the regression equation because of
insufficient variation in the data regarding that variable. An example
also provided the opportunity to emphasize the need for checking the
second order conditions. Furthermore it was suggested that when a
variable drops out from a regression equation other methods such as surveys

should be used in addition to regression analysis in order to obtain

an estimate of the effect of that variable on sales. Thirdly, the

importance was emphasized of exploring and reporting the sensitivity

of profit to changes in the decision variables. And finally it was

argued that in using the equality of marginal revenue and marginal cost

as an optimization rule, opportunity cost should be included in marginal

cost.

The difficulties discussed in this paper are not meant to dis-

courage the application of the Dorfman-Steiner theorem in real situations.

Quite on the contrary, the comments presented here should be considered

as warnings and as suggestions to assist in making the economic inter-

pretation of these applications more meaningful.

TABLE 1

Sales and Profit for different levels of
Price and advertising

p	s	q	π
27	100	92.04	1,832.86
27	150	96.01	1,866.21
27	200	98.97	1,878.37
27	210	99.55	1,880.55*
27	220	99.86	1,877.06
27	270	101.97	1,871.37
27	350	104.56	1,845.76
27	450	107.07	1,798.47
27	600	109.95	1,708.95
27	700	111.49	1,641.29
36	100	74.05	2,121.5
36	150	78.09	2,192.7
36	200	80.97	2,229.1
36	270	83.97	2,249.1
36	300	85.02	2,250.6*
36	350	86.56	2,246.8
36	450	87.90	2,237.0
36	500	90.13	2,203.9
36	600	91.95	2,158.5
36	700	93.49	2,104.7

T A B L E 1 (cont.)

p	s	q	π
42.77	100	60.50	2,124.58
42.77	150	64.55	2,223.50
42.77	200	67.43	2,279.40
42.77	300	71.48	2,328.32
42.77	350	73.02	2,334.94
42.77	367.7	73.53	2,335.87**
42.77	385	73.98	2,335.24
42.77	450	75.53	2,327.24
42.77	500	76.59	2,316.21
42.77	600	78.41	2,282.92
42.77	700	79.95	2,239.76
50	100	46.05	1,926.20
50	150	50.09	2,053.96
50	200	52.97	2,130.68
50	300	57.02	2,208.88
50	400	59.90	2,235.60
50	440	60.85	2,237.40*
50	500	62.13	2,232.62
50	600	63.95	2,213.80
50	700	65.49	2,181.56
60	100	26.04	1,306.10
60	150	30.09	1,474.86

T A B L E   1   (cont.)

p	s	q	π
60	200	32.97	1,580.38
60	300	37.02	1,699.08
60	385	39.52	1,749.08
60	450	41.07	1,767.78
60	500	42.13	1,774.92
60	540	42.90	1,776.60*
60	600	43.95	1,773.30
60	700	45.49	1,756.46

---

*maximum profit corresponding to optimal advertising for a given price

**maximum maximorum (optimal advertising, optimal price)

T A B L E   2

Sensitivity of Profit to Advertising

| | Advertising Range | | | |
Price	profit 1% below optimum		profit 5% below optimum	
27	-34%	+48%	-70%	+127%
36	-33%	+46%	-63%	+114%
42.77	-28%	+42%	-60%	+101%
50	-27%	+36%	-56%	+ 95%
60	-25%	+26%	-47%	+ 70%

T A B L E   3

Maximum profit for Different Prices

p	s	q	π
20	140	109.40	1,391.60
27	210	99.55	1,880.55
36	300	85.02	2,250.60
40	340	78.27	2,321.18
42.77	367.7	73.53	2,335.87
45	390	69.60	2,324.40
50	440	60.85	2,237.40
60	540	42.90	1,776.60
66	600	31.95	1,317.00
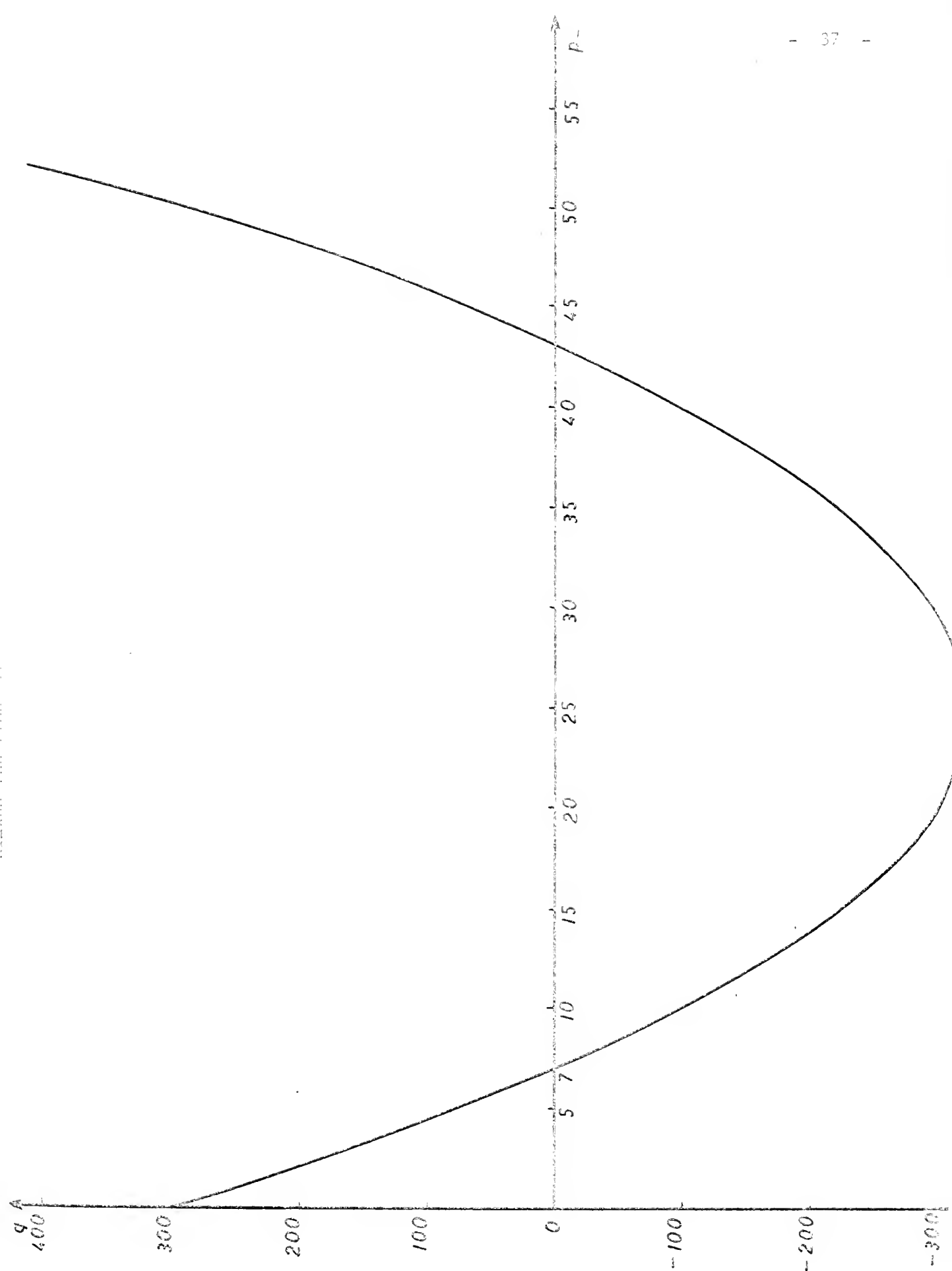
Figure 2

S as a function of advertising
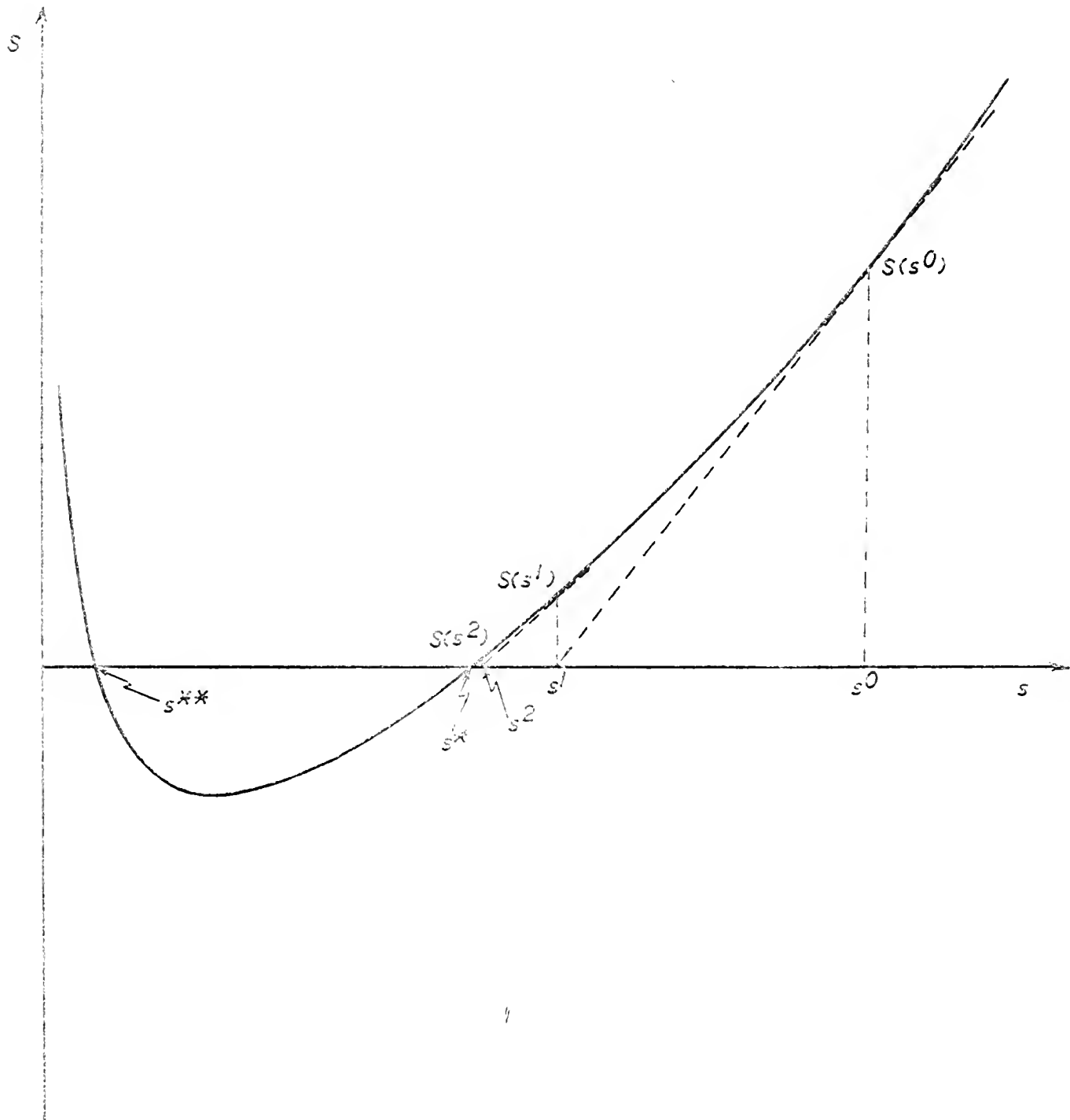
Figure 3

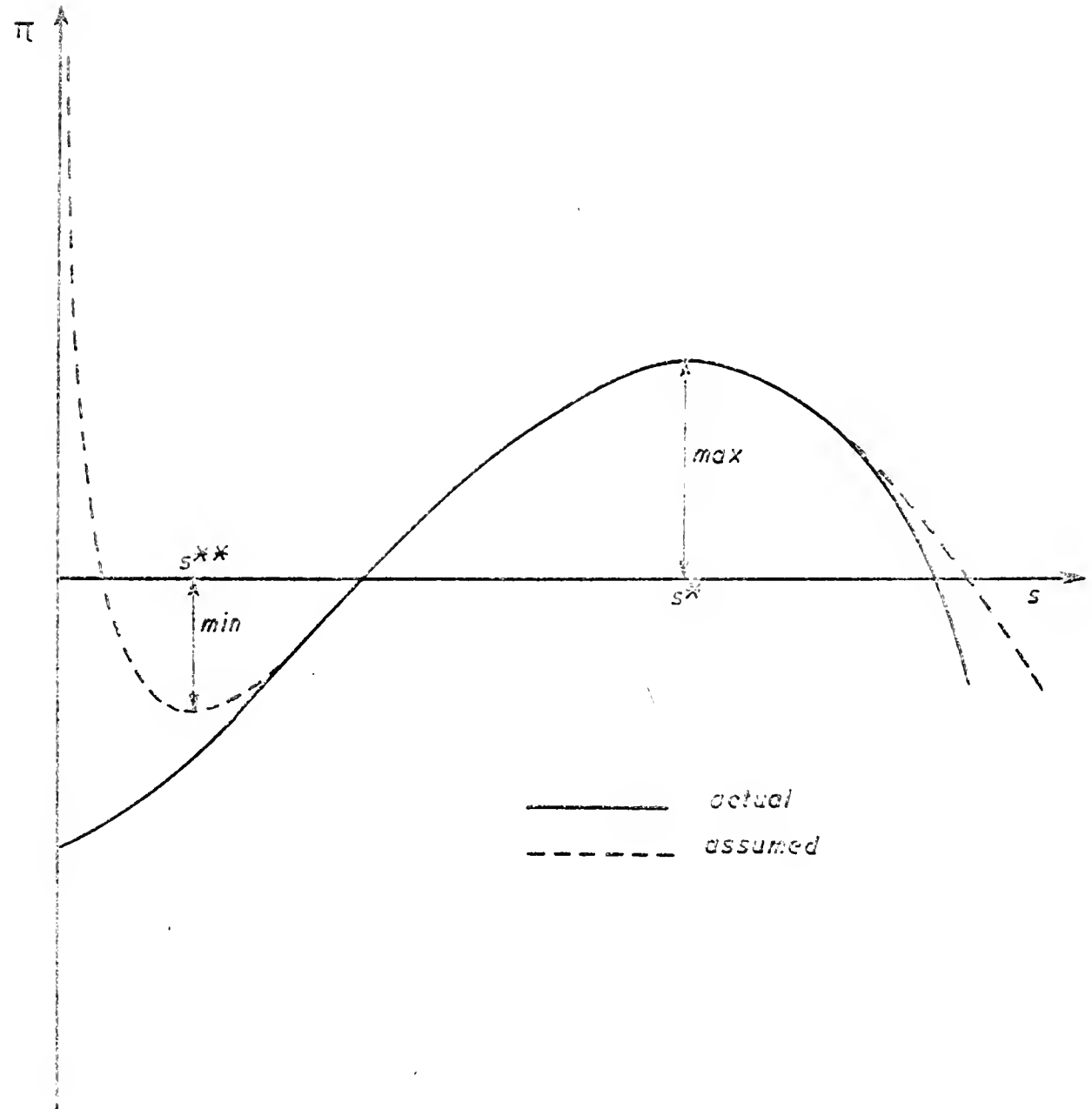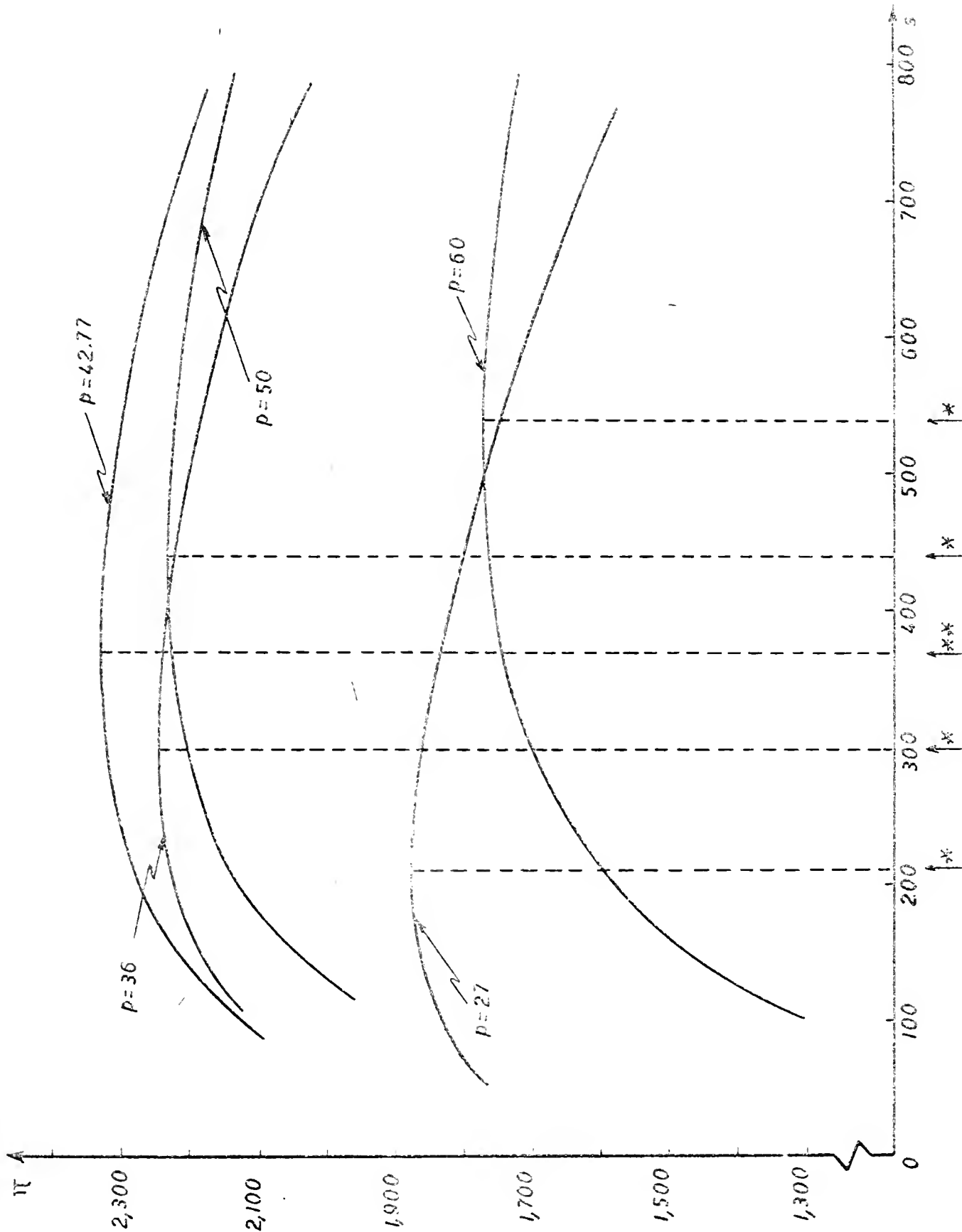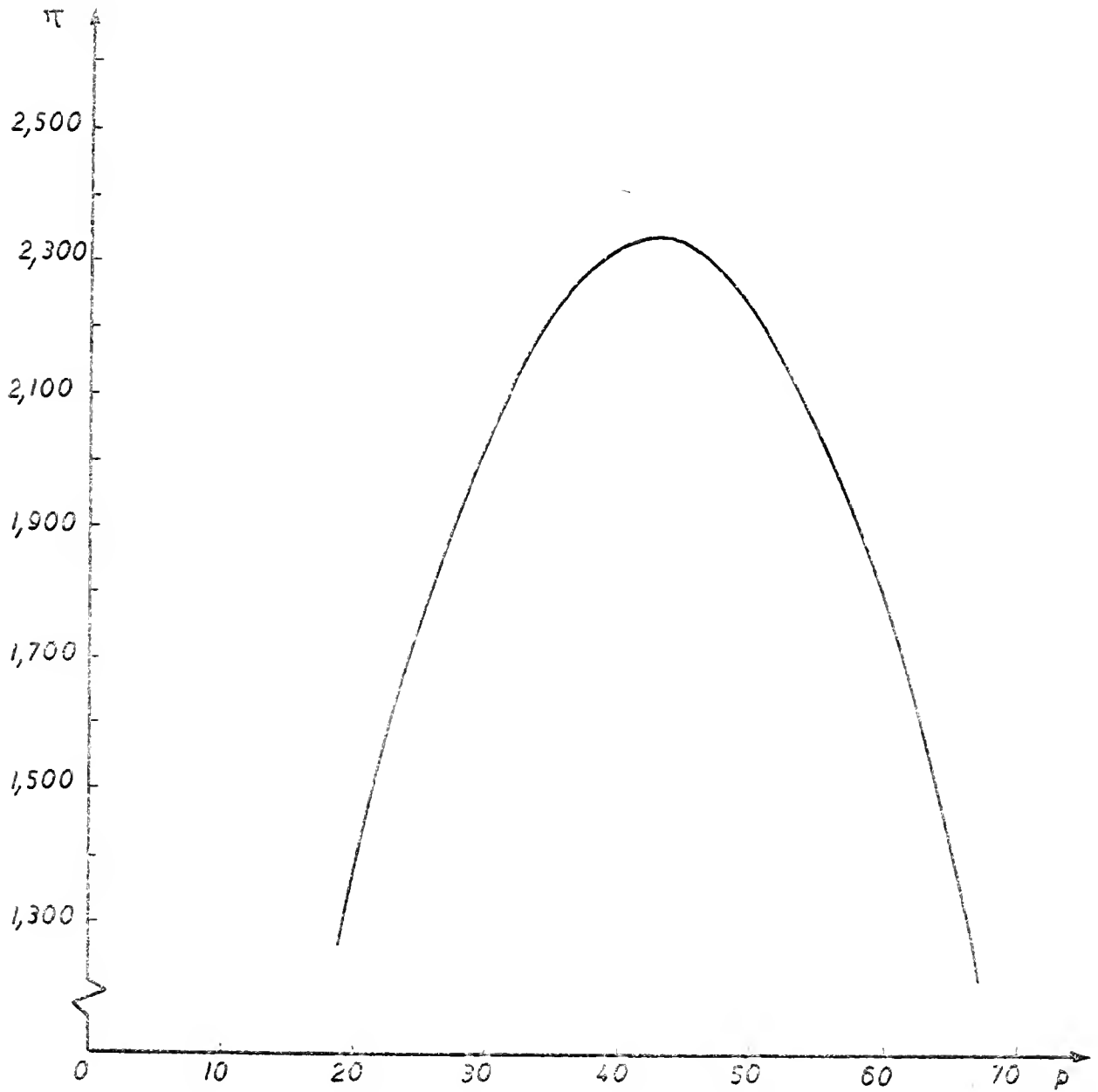Sales (associated with advertising) as a
function of advertising
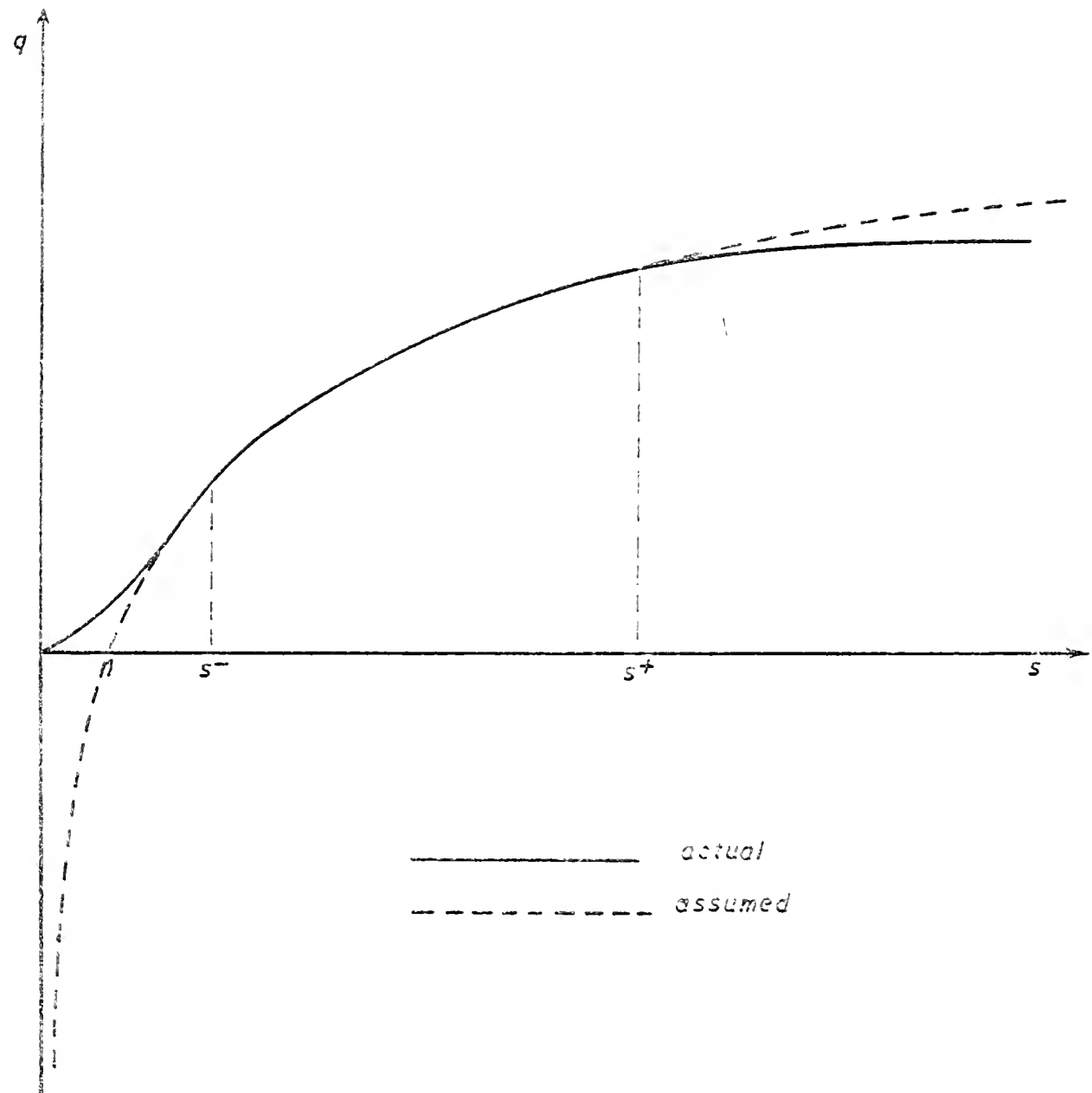
Figure 4

Profit as a function of advertising
(optimum price)

Figure 6

Profit as a function of price
(with associated optimum advertising)

Footnotes

[1] See in particular the work by Palda [12], [15], [16], and by Lambin [8], [9], [10], [11], [12].

[2] We refer to work by Bass [1], Bass and Parsons [2], Cook and Rahman [4] and Beckwith [3]. See also Melrose [13] for a study of the effect os sales on advertising.

[3] Errors in specification result in biased estimates. See Goldberger [6, pp. 196-7].

[4] See the references in footnote 1. On distributed lags see Koyck [7].

[5] For a derivation, see the mathematical appendix to Dorfman and Steiner's paper by Bass, Buzzell et al. [5, p. 217].

[6] Most authors refer to the mathematical appendix by Bass, Buzzell et al. [5, p. 214-5] for a proof that the second order conditions hold. However, that proof is incorrect and it is not obvious under what general conditions the Hessian determinant will be negative definite. We will encounter several situations where it is necessary to check the second order conditions.

[7] $p = \bar{p}$ and $q = \bar{q}$ is a point on the regression line.

[8] Of course it is also possible that n_p is constant in which case relation (7) can be used as it stands. The third example in this section will be a case with constant price elasticity.

[9] By saying 'about 20 per cent' rather than 'exactly 20 per cent' we recognize the fact that $a = 100$ and $b = 2$ are estimated values.

[10] An implicit assumption in (14) is that there are no lagged effects of advertising. We do not need to introduce lags here because the purpose is to demonstrate the interdependency of the optimal values of both decision variables. All logarithms are assumed to be natural logarithms unless otherwise stated.

Footnotes

[11]We could easily generalize to allow any starting value for s, but it is not necessary.

[12]i in s^i is a superscript, not an exponent.

[13]ε and δ depend on the degree of accuracy we want.

[14]An example where price has the wrong sign due to lack of variation in the price data is reported by Lambin [10].

[15]Note that we assumed a priori that demand is a linear function of price. Other functions are of course possible.

[16]Recall that we are talking about percentage deviations from the optimum, not absolute deviations.

[17]Of course it is nice to know both.

[18]So, not only is the price level assumed optimal, but also the distribution variable.

[19]$\partial q/\partial s = 0.303$ is based on the short term effect of advertising. Long term considerations will be added later in this section.

[20]For the argument on which this relationship is based, see Palda [15, p. 17] [16, p. 191].

[21]Not to be confused with marginal revenue product of advertising $p \partial q/\partial s$.

[22]One has to be careful here, so that an opportunity cost is not already incorporated in the discount rate.

References

1. F. M. Bass, "A Simultaneous Equation Regression Study of Advertising and Sales on Cigarettes", Journal of Marketing Research, Vol. 6, No. 3 (August 1969), pp. 291-300.

2. F. M. Bass and L. J. Parsons, "Simultaneous Equation Regression Analysis of Sales and Advertising", Applied Economics, Vol. 1, No. 2 (May 1969), pp. 103-124.

3. N. Beckwith, The Response of Competing Brands to Advertising: A Multivariate Regression Test, Unpublished Ph.D. Dissertation, Purdue University, 1970.

4. V. J. Cook and A.N.M. Rahman, "Emark: An Analytic Demand Forecasting Model", Working Paper P-43-9, Marketing Science Institute (May 1968).

5. R. Dorfman and P. O. Steiner, "Optimal Advertising and Optimal Quality", in F.M. Bass, R.D. Buzzell et al. (eds.), Mathematical Models and Methods in Marketing, Homewood, Ill.: R.D. Irwin, 1961, pp. 203-219.

6. A. S. Goldberger, Econometric Theory, New York, N. Y.: Wiley, 1964.

7. L. J. Koyck, Distributed Lags and Investment Analysis, Amsterdam: North-Holland, 1954.

8. J. J. Lambin, "Optimal Allocation of Competitive Marketing Efforts", Working paper No. 1, CESAM, Université Catholique de Louvain, September 1968.

9. J.J. Lambin, "Advertising and Competitive Behavior: The Case of a Consumer Durable Goods", Working paper No. 2, CESAM, Université Catholique de Louvain, February 1969.

10. J. J. Lambin, "Measuring the Profitability of Advertising: An Empirical Study", Journal of Industrial Economics, Vol. 17, No. 2 (April 1969). pp. 86-103.

11. J. J. Lambin, Marketing Models: An Econometric Approach, Scott, Foresman and Cy. (forthcoming).

12. J. J. Lambin and K. S. Palda, "Parameter Estimates for Optimal Marketing Decisions", in R. L. King, Marketing and the New Science of Planning, Chicago: American Marketing Association, 1968, pp. 69-74.

13. K. B. Melrose, "An Empirical Study on Optimizing Advertising Policy", Journal of Business, Vol. 42, No. 3 (July 1969), pp. 282-292.

14. D. B. Montgomery and A. J. Silk, "Distributed Lag Models of Response to a Communications Mix", Working paper No. 470-70, Alfred P. Sloan School of Management, M.I.T., July 1970.

15. K.S. Palda, The Measurement of Cumulative Advertising Effects, Englewood Cliffs, N.J.: Prentice-Hall, 1963

16. K. S.Palda, Economic Analysis for Marketing Decisions, Englewood Cliffs, N.J.: Prentice-Hall, 1969.

Date Due